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Soliton ratchets

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The mechanism underlying the soliton ratchet, both in absence and in presence of noise, is investigated. We show the existence of an asymmetric internal mode on the soliton profile that couples, through the damping in the system, to the soliton translational mode. Effective soliton transport is achieved when the internal mode and the external force are phase locked. We use as a working model a generalized double sine-Gordon equation. The phenomenon is expected to be valid for generic soliton systems.

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One of the phenomena which is presently attracting interest both in physics and in biology is the so-called ratchet effect [1]. In simple terms, a ratchet system can be described as a periodically forced Brownian particle moving in an asymmetric potential in the presence of damping and periodic driving. The periodic forcing keeps the system out of equilibrium so that the thermal energy can assist the conversion of the ac driver into effective work (direct motion of the particle) without any conflict with the second law of thermodynamics. This phenomenon has been found in several physical [2] and biological [3] systems and is presently considered as a possible mechanism by which biological motors perform their functions [4]. For ode systems with damping, additive forcing, and noise, the ratchet effect can be viewed as a phase-locking phenomenon between the motion of the particle in the periodic potential and the external driver [5]. Ratchet dynamics have also been observed in more complicated systems such as overdamped ϕ^4 models [6], chains of coupled particles with degenerate on-site potentials [7], long Josephson junctions with modulated widths [8], and inhomogeneous parallel Josephson arrays [9], three-dimensional helical models [10], etc. These are infinite dimensional systems described by continuous or discrete equations of soliton type, with asymmetric potentials, damping, and periodic forcing, in which the ratchet phenomenon manifests as unidirectional motion of the soliton (soliton ratchet). For overdamped systems one can reduce the soliton ratchet to the usual single-particle ratchet by using a collective coordinate approach for the center-of-mass of the soliton [6]. For underdamped or moderately damped systems, however, this could be inappropriate, since the radiation field present in the system can play an important role for the generation of the phenomenon.

The aim of this article is to investigate the mechanism underlying soliton ratchets both in the absence and in the presence of noise. To this end we use an asymmetric double sine-Gordon equation as a working model for studying the effect (the phenomena, however, will not depend on the particular model used). We show that the asymmetry of the potential induces a spatially asymmetric internal mode on the soliton profile which can be excited by the periodic force. In the presence of damping, this mode can exchange energy with the translational mode so that the soliton can have a net motion under the action of the ac force. In this mechanism, the damping plays the role of coupling between the internal mode and the translational mode of the soliton. We find that, for fixed amplitude and frequency of the ac force, there is an optimal value of the damping for which the transport (i.e., the velocity achieved by the soliton) becomes maximal. In this case the frequency of the internal mode and the one of the external driver, perfectly match (phase locking). A similar resonant behavior is also observed by varying the frequency of the ac force, keeping fixed the other parameters of the system. At very high damping and fixed amplitude of the forcing, the internal mode oscillation becomes very small, and the transport due to the soliton ratchet is strongly reduced. At low damping and higher forcing we find, quite surprisingly, that current reversals can occur. Finally, we show that soliton ratchets can survive the presence of noise in the system.

We start by introducing the following generalized double sine-Gordon equation:

$$\phi_{tt}-\phi_{xx}=-\sin(\phi)-\lambda\sin(2\phi+\theta)\equiv-\frac{dU(\phi)}{d\phi} \qquad (1)$$

with the potential $U(\phi)=C-\cos(\phi)-(\lambda/2)\cos(2\phi+\theta)$. Here λ is the asymmetry parameter, θ is a fixed phase, and C a constant that fixs the zero of the potential. A discrete version of this equation was introduced in Ref. [11] in terms of a chain of elastically coupled double pendula assembled by a gear of ratio 1/2 with a phase angle θ between them. For $\lambda=0$, Eq. (1) gives the well known sine-Gordon equation (SGE) with exact soliton solutions, while for $\lambda\neq 0$ and $\theta=0$ (mod π) it reduces to the proper double sine-Gordon equation (note that in both cases the potential is periodic and symmetric in ϕ). In the following discussion we are interested in the case $\theta\neq 0$ (mod π) for which the periodic potential becomes asymmetric. We shall refer to this case as the asymmetric double sine-Gordon equation (ADSGE). In particular, we fix $\theta=\pi/2$ in Eq. (1) in order to have maximal asymmetry, and choose $C=\cos(\phi_0)-\lambda/2\sin(2\phi_0)$, with $\phi_0=\arcsin[(1-A)/4\lambda]+2n\pi$ and $A=\sqrt{1+8\lambda^2}$, to have the zero of the potential in correspondence with its absolute minima ϕ_0. We also assume, for simplicity, $\lambda\in[-1,1]$ to avoid relative minima appearing in the potential. Besides the mentioned mechanical model, Eq. (1) is also linked to another interesting physical system i.e., a one-dimensional array of inductively coupled Superconducting quantum interference devices (SQUID's), each consisting of a loop of a Josephson junction in parallel with a series of two identical Josephson junctions. The single element of this array was studied in Ref. [12] in which it was shown that, due to the ratchet effect, the system can rectify periodic signals. In analogy with the single-particle ratchet, it is reasonable to introduce in the distributed model, periodic forcing, damping, and noise, leading to the following perturbed ADSGE:

$$\phi_{tt}-\phi_{xx}+\sin(\phi)+\lambda\cos(2\phi)=-\alpha\phi_t+\epsilon\sin(\omega t+\theta_0)+n(x,t). \tag{2}$$

Here α denotes the damping constant, $n(x,t)$ is white noise with autocorrelation

$$\langle n(x,t)n(x',t')\rangle=D\delta(x-x')\delta(t-t'), \tag{3}$$

and ϵ, ω, θ_0 are, respectively, the amplitude the frequency and the phase of the driver. Traveling wave solutions of the unperturbed ADSGE, i.e., solutions which depend on $\xi\equiv(x-Vt)/\sqrt{1-V^2}$ [note that Eq. (1) is Lorentz invariant] can be found by substituting $\phi\equiv\phi(\xi)$ in the left-hand side of Eq. (2) and equating it to zero. After one integration in ξ we obtain

$$\int_{\phi_0}^{\phi}\frac{d\phi}{\sqrt{2(E+U)}}=\xi-\xi_0, \tag{4}$$

which gives, after the inversion of the integral at $E=0$ (top of the reversed potential), the 2π-kink (antikink) solutions as

$$\phi_K^{\pm}=\phi_0+2\tan^{-1}\left\{\frac{\text{sign}(\lambda)AB}{A-1-\eta\sinh\left[\pm\frac{\xi}{2}\sqrt{AB/|\lambda|}\right]}\right\}, \tag{5}$$

where $\eta=2\lambda\sqrt{2(1+A)}$, and $B=\sqrt{2(4\lambda^2-1+A)}$ (the plus and minus signs refer to the kink and antikink solutions, respectively). Note that in the limit $\lambda\to 0$, Eq. (5) reduces to the well-known soliton solution of SGE. To investigate the existence of internal modes in the system, we linearize the ADSGE around the solution in Eq. (5), i.e., we look for solutions of the form $\phi=\phi_K^{\pm}+\psi$ with

$$\psi(x,t)=\exp(i\omega t)f(x), \quad f(x)\ll 1.$$

This leads to the following eigenvalue problem on the whole line

$$f_{xx}+\{\omega^2-\cos[\phi_K^{\pm}]+2\lambda\sin(2\phi_K^{\pm})\}f=0, \tag{6}$$

with $f_x(\pm\infty)=0$, which can be easily solved by numerical methods for any finite length of the system. In Fig. 1 we report the numerical spectrum of Eq. (6) as a function of λ. We see that except for the sine-Gordon limit $(\lambda=0)$, there is an internal mode frequency Ω_i below the spectrum of the phonon band (the zero mode, existing for all values of λ, is not plotted for graphical convenience). In the inset of the figure the shape of the internal mode is also reported, from which we see that the asymmetry of the potential induces a spatial asymmetry in $f(x)$. In the presence of a periodic force this internal mode can be easily excited.

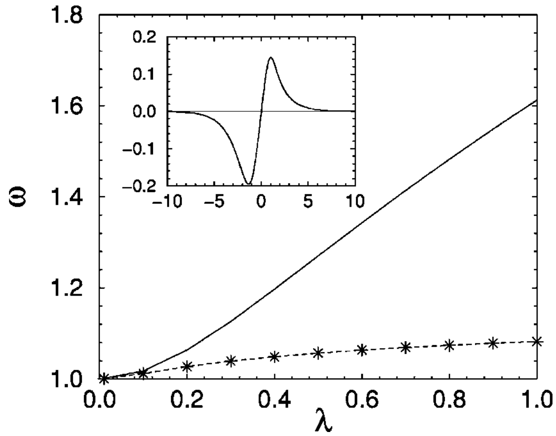

FIG. 1. Small oscillation spectrum versus λ. Above the solid line are the phonon's modes. The stars joined by the dashed line refers to the internal mode. The inset shows the spatial profile of the internal mode at $\lambda=0.5$.

To understand the role of the various elements of the problem, (i.e., asymmetry of the potential, internal mode, damping, forcing, and noise) it is better to consider first the zero noise case (deterministic soliton ratchet). By viewing the soliton as a string lying on the potential surface $S(U,\phi,x)$ and connecting adjacent minima, the following picture of the phenomenon can be given. If the potential is asymmetric (in ϕ) the transition from the top of the potential to one minimum and from the top of the potential to the other minimum, is also asymmetric (it will be more rapid for the part of the string lying on the region where the potential is more stiff). Thus, the potential asymmetry in ϕ induces an asymmetry in space which can be seen both in the 2π-kink profile and in the internal mode. In presence of an ac force, but in the absence of damping, this asymmetry will not produce transport, i.e., the string will slide, without any friction, back and forth on the potential profile along the x direction. The presence of damping, however, introduces friction in this sliding, and the part of the string moving on the stiff part of the potential profile dissipates more that the other. This asymmetry in the dissipation produces net motion for the soliton (the string moves in the direction in which it approaches the potential minimum more smoothly). We can say that the effect of the damping is to couple the internal mode to the translational mode. The mechanism underlying the deterministic soliton ratchet can then be described as follows: the ac force pumps energy in the internal mode which is converted into net dc motion by the coupling with the zero mode induced by the damping. From this picture one can easily predict that in the absence of the internal mode, or in the absence of damping, no soliton ratchet can exist. Moreover, one expects that the maximal effect in transport, is

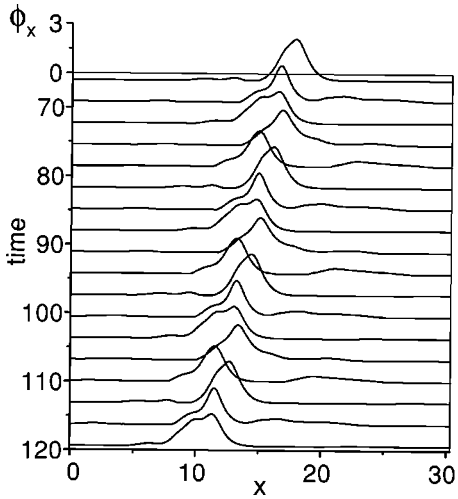

FIG. 2. Time evolution of the x derivative of the kink profile while executing ratchet dynamics. The parameter values are $\lambda=0.5$, $\theta_0=\pi/2$, $\alpha=0.6$, $\epsilon=0.8$, $\omega=0.8$, and $D=0$.

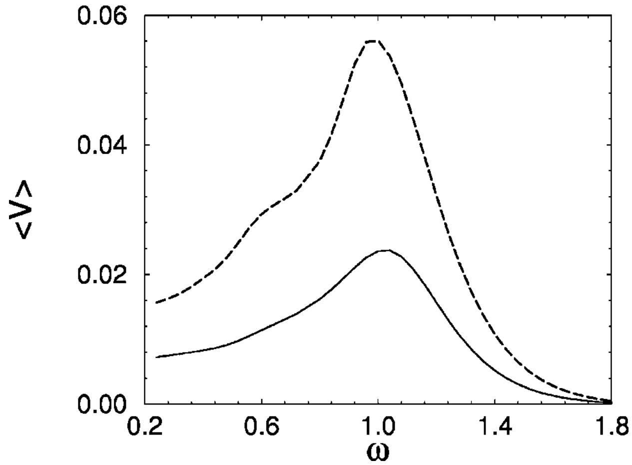

FIG. 3. Mean velocity of the kink center-of-mass versus the frequency of the ac force for $\epsilon=0.4$ (solid line) and $\epsilon=0.6$ (dashed line). The other parameters are fixed as in Fig. 2 except for the asymmetry parameter, which is $\lambda=-0.5$.

observed when the internal oscillation and the external force are synchronized (phase locked).

In order to confirm this picture, we have performed direct numerical integrations of the ADSGE for different values of the system parameters. First, we have checked that in the SGE limit, i.e., when $\lambda=0$, the ratchet dynamics does not exists. This agrees with the fact that in this case there is no asymmetry in the potential and no internal mode. At this point we remark that the dynamics of a SG kink subject to a periodic force was also investigated in Ref. [13], in which it was shown that in the absence of damping the kink can acquire a finite velocity depending on the initial phase of the ac force. Net motion of a SGE soliton was shown to be also possible in the presence of a small damping, if the ac force excites a phonon mode that exchanges energy with the soliton [14]. These cases, however, should not be confused with soliton ratchets since they strongly depend on initial conditions (if one average on initial conditions the transport disappears). Moreover, in contrast with soliton ratchets, these effects exist only at zero or at very low damping. Second, we have checked that for $\lambda\neq 0$ (asymmetric potential) but in the absence of damping, soliton ratchets also do not exist (for brevity we will not expand on these cases here). From this analysis we conclude that, in analogy with the deterministic single-particle ratchets, the asymmetry of the potential, the damping and, obviously the periodic forcing, are crucial ingredients for soliton ratchets. In Fig. 2 we show a prospectic view of the soliton ratchet dynamics as obtained from numerical integration of Eq. (2). We remark that the direction of the motion is fixed by the asymmetry of the potential and can be inverted by changing the sign of λ. We see that, except for the soliton profile which is wobbling, no phonons are present in the system. This confirms the relevance of the internal mode in the phenomenon. We also find that for some parameter values, the net motion is more effective. To investigate the dependence of the phenomenon on parameters, we have performed numerical simulations of Eq. (2) both by fixing all parameters and changing ω, and by fixing all parameters and changing α. In Fig. 3 we show the average velocity (computed by using an integration time $t=1000$) of the soliton center-of-mass versus the frequency of the external driver, for two different values of the amplitude of the ac force (the low damping part of the curves was not computed due to the longer integration times required in this case). From this figure we see that $\langle V\rangle$ has a maximum at $\omega\sim 1$ (i.e., $\omega=1.04$ and $\omega=1$ for $\epsilon=0.4$ and $\epsilon=0.6$, respectively) which is very close to the internal mode frequency $\Omega_i\approx 1.0562$ (the discrepancy is within the numerical accuracy of our numerical scheme). By increasing the amplitude of the forcing, the dynamics gets more complicated (breatherlike excitations can appear) and the resonance peak in frequency more pronounced.

A similar resonant behavior is expected to also exist as a function of the damping. When the damping is very high, indeed, the internal mode is almost suppressed by the damp-

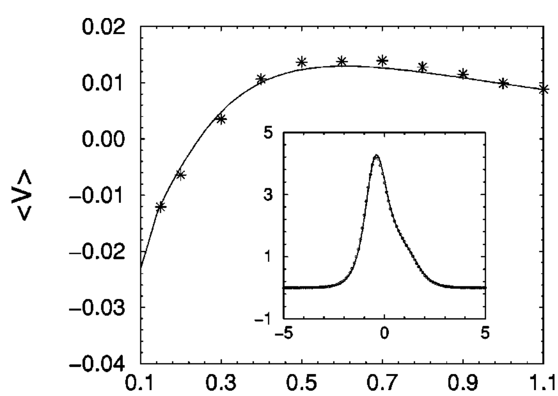

FIG. 4. Mean velocity of the center-of-mass of the kink versus α, for parameter values $\lambda=-0.5$, $\theta_0=\pi/2$, $\omega=0.4$, and $\epsilon=0.5$. The continuous curve refers to the case $D=0$, while the stars refer to the nondeterministic case $D=0.01$. The continuous and dotted curves in the inset refer to two soliton ϕ_x profiles at $D=0$, taken in the comoving frame and separated in time by one period of the driver.

ing, while when it is very low, the coupling between the internal mode and the translational mode is very small, both cases giving minimal transport. In between these extremes, a value of the damping which allows the internal mode to synchronize with the external driver and optimize transport, should then exist. This is what we observe in Fig. 4 where the velocity vs damping is reported as a continuous curve. To show that the internal mode is phase locked with the external driver, we have plotted in the inset of this figure the ϕ_x profiles in the comoving frame (i.e., the drift motion was subtracted) at two fixed times $t_1=211.6$ (solid line) and $t_2=227.3$ (circles) separated by one period $T=2\pi/\omega=15.7$ of the driver. We see that the profiles overlap each other, i.e., the oscillation on the kink profile is perfectly synchronized with the external driver (phase locking). From Fig. 4 we also see, quite surprisingly, that at low damping current reversals can occur (note that the average velocity becomes negative for α less than $\alpha_{cr}\sim 0.24$).

We find that the value of α_{cr} increases as the amplitude of the driver is increased. The occurrence of this phenomenon, which resembles the one observed in single-particle ratchets at low damping [5,15], seems to be related more to the phonon-soliton interaction than to the internal mode mechanism described above (at low dampings a complicate transferral of energy between phonons, internal mode, and translational mode, can arise). To understand this phenomenon, however, a more detailed study is required.

Finally, we have investigated the effect of the noise on deterministic soliton ratchets. A preliminary analysis shows that for low noise amplitudes the effect of the noise on the phenomenon is minimal, in the sense that the dynamics gets dressed by the noise, but after averaging on the noise, almost the same soliton mean velocity $\langle V \rangle$ is obtained. This is shown by the stars in Fig. 4, which represent the numerical values of $\langle V \rangle$ calculated in the presence of a noise of amplitude $D=0.01$. The fact that soliton ratchets can survive the presence of weak amplitude noise can be understood as a consequence of the structural stability of phase-locking phenomena against small fluctuations. This indicates that the phenomenon can also exist in real systems such as one-dimensional arrays of inductively coupled SQUID's. We finally remark that the presented mechanism of soliton ratchets is expected to be also valid for other soliton systems.

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